Topics in complex analysis. Lecture 13 (Dec 16, 2024)

Theorem 8.19 (Hartogs' theorem on holomorphy). *Let $f : U \to \mathbb{C}$ be holomorphic separately in each variable. Then f is holomorphic in the sense of Definition 8.1.*

Proof: We claim f can be locally written as a convergent power series: $\forall z_0 \in U \ \exists r > 0$ st

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z - z_0)^\alpha \qquad \forall z \in D_r^n(z_0)$$

The holomorphy of f will follow from differentiation thms. for convergent power series.

WLOG (up to a translation) we may and will assume $z_0 = 0$. Choose $r > 0$ st $D_{2r}^n(0) \subseteq U$.

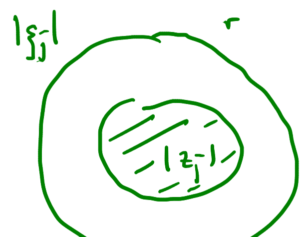
Step 1: Suppose we already know f is locally bounded.

This means f is bounded on $D_r^n(0)$. Notice then 8.5 carries over to separately holo fct by iteration of the 1-dim Cauchy integral formula. This gives $\forall z \in D_r^n(0)$

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1| = r} \cdots \int_{|\xi_n| = r} \frac{f(\xi)}{\prod_{j=1}^{n} (\xi_j - z_j)} \, d\xi_1 \cdots d\xi_n$$

For $|z_j| < |\xi_j|$ using geometric series we get

$$\frac{1}{\xi_j - z_j} = \sum_{n_j = 0}^{\infty} \frac{z_j^{n_j}}{\xi_j^{n_j + 1}}$$



This sum converges uniformly in $\xi_j \in \partial B_r^1(0)$ and $z_j \in B_{r_0}^1(0)$ where $r_0 < r$ is fixed.

Since f is bounded on $\overline{D}_r^n(0)$, using loc unif cvg shown above, we can swap integration and summation:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} f(\zeta) \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{z_1^{m_1} \cdots z_n^{m_n}}{\zeta_1^{m_1+1} \cdots \zeta_n^{m_n+1}} d\zeta_n \cdots d\zeta_1$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha, \qquad c_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} \frac{f(\zeta)}{\zeta^{\alpha+(1,\dots,1)}} d\zeta_n \cdots d\zeta_1$$

And the series converges loc unif on $D_r^n(0)$

We will prove Hartog's thm by induction over n.

The case $n=1$ is clear.

Induction hypothesis: f is jointly holo in the first $n-1$ variables and holo in the last variable.

<u>Step 2</u> We claim f is locally bdd on a smaller polydisk: $\exists$ closed disks $E_j \subseteq \overline{B}_r^1(0) \subseteq \mathbb{C}$ (not necessarily containing 0!) with nonempty interior and $E_n = \overline{B}_r^1(0)$ st f is bdd on $E_1 \times \dots \times E_n$.

Given $N \in \mathbb{N}$ we define

$$\Omega_N := \left\{ z' \in \prod_{j=1}^{n-1} \overline{B}_r^1(0) : \sup_{z_n \in \overline{B}_r(0)} |f(z', z_n)| \leq N \right\}.$$

By induction hypoth, $\forall z_n \in \overline{B}_r(0)$, the function $z' \longmapsto f(z', z_n)$ is continuous, hence

$$z' \longmapsto \sup_{z_n \in \overline{B}_r(0)} |f(z', z_n)| \text{ is lsc.}$$

By lsc, Ω_N is closed. On the other hand $\forall z' \in \prod_{j=1}^{n-1} \overline{B_r(0)}$ the function $z_n \longmapsto f(z', z_n)$ is holo on $B_{2r}(0)$, hence bdd on $\overline{B_r(0)}$. Hence

$$\underline{\prod_{j=1}^{n-1} \overline{B_r(0)}} = \bigcup_{N \in \mathbb{N}} \Omega_N$$

The lhs is a set with nonempty interior which can be exhausted by countably many closed sets Ω_N. The Baire category thm implies $\exists N \in \mathbb{N}$: Ω_N has nonempty interior. In part Ω_N contains a closed polydisk with nonempty int. By def of Ω_N, f is bdd on this polydisk.

Step 3 If $f: D_r^n(z_0) \longrightarrow \mathbb{C}$ is separately holo in $\underline{z'}$ and z_n, and bdd in a polydisk $D^n_{\underbrace{(r'_1, \dots, r'_1}_{n-1 \text{ radii} < r}, r)}(z_0)$

first $n-1$ coord

then f is holo on $D_r^n(z_0)$.

For simplicity we assume $z_0 = 0$. By separate holo for $(z', z_n) \in D_r^n(0)$ we can write

$$f(z', z_n) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} c_\alpha(z_n)(z')^\alpha \qquad (*)$$

loc unif wrt. z', and

$$c_\alpha(z_n) = \frac{\partial^\alpha}{\partial (z')^\alpha} \frac{f(0, z_n)}{\alpha!}$$

By Cor 8.6 (Cauchy formula for derivatives)

$$c_\alpha(z_n) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r'/2} \cdots \int_{|\zeta_{n-1}| = r'/2} \frac{f(\zeta', z_n)}{\zeta_1^{\alpha_1+1} \cdots \zeta_{n-1}^{\alpha_{n-1}+1}} \, d\zeta'.$$

Indeed c_α is differentiable in $z_n \in B_r(0)$: since f is hol, a theorem on the differentiation of parameter-integrals yields the claim. In turn

$$v_\alpha(z_n) := \frac{1}{|\alpha|} \log |c_\alpha(z_n)|$$

is subharmonic.

We want to apply Lemma 8.18 in order to control f locally on the larger polydisk. By unif cvg of $(*)$ wrt $|z'| \leq r_2$ where $r_2 < r$ we obtain

$$\lim_{|\alpha| \to \infty} |c_\alpha(z_n)| \, r_2^{|\alpha|} = 0 \quad \forall z_n \in B_r(0)$$

≤ 1

which gives

$$\limsup_{|\alpha| \to \infty} v_\alpha(z_n) \leq -\log(r_2)$$

By the estimate for Cauchy integrals

$$|c_\alpha(z_n)| \leq \frac{\sup_{z' \in D^{n-1}_{r'/2}(0)} |f(z', z_n)|}{\left(\frac{r'}{2}\right)^{|\alpha|}} \leq \frac{B}{\left(\frac{r'}{2}\right)^{|\alpha|}}$$

>1

Taking logarithm of this yields

$$v_\alpha(z_n) \leq \frac{1}{|\alpha|} \log B - \log \frac{r'}{2} \leq \log B - \log \frac{r'}{2}$$

$$\Longrightarrow \sup_{\alpha \in \mathbb{N}_0^{n-1}} v_\alpha(z_n) < \infty$$

Applying Lemma 8.18 we get: $\forall\, 0 < r_1 < r_2$ $\exists\, N \in \mathbb{N}$ $\forall\, |\alpha| \geq N$ and all $z_n \in B_{r_1}(0)$:

$$v_\alpha(z_n) \leq -\log r_1 \Longrightarrow |c_\alpha(z_n)| \leq r_1^{-|\alpha|}$$

$$\forall\, z_n \in B_{r_1}(0)$$

In particular the sum

$$f(z', z_n) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} c_\alpha(z_n)(z')^\alpha$$

converges unif on $\overline{D^n_{r_0}(0)}$ $\forall\, 0 < r_0 < r_1$ (exercises). In part f is bdd, hence holo on $D^n_{r_0}(0)$ by Step 1. Since the radii were arbitrary, f is holo on $D^n_r(0)$.

Step 4 By Step 2, f is bdd on a closed polydisk $\overline{D_{r'}^{n-1}}(z_0) \times \overline{B_r}(0)$. In addition we know $(z_0, 0) \in D^n_r(0)$ by construction/ $\overline{D_{r'}^{n-1}}(z_0) \times \overline{B_r}(0) \subseteq D^n_r(0)$. By Step 3, f is holo on $\left(D_r^{n-1}(z_0) \times B_r(0)\right) \cap D^n_r(0)$

Since $z_0 \in D_r^{n-1}(0)$ we have

$$(0', 0) \in \left(D_r^{n-1}(z_0) \times B_r(0)\right) \cap D^n_r(0).$$

$\square$